LQ dynamic games

as receding-horizon variational inequalities

Emilio Benenati   KTH Stockholm

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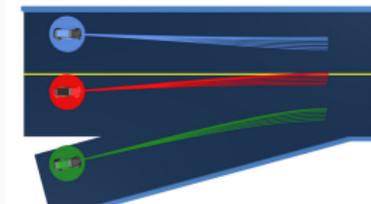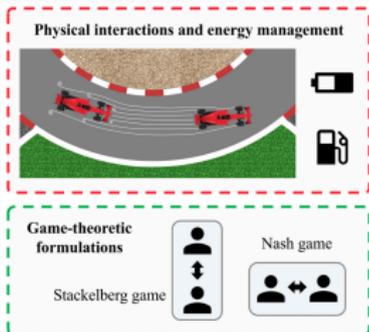Giuseppe Belgioioso   KTH Stockholm

December 9, 2025
CDC '25
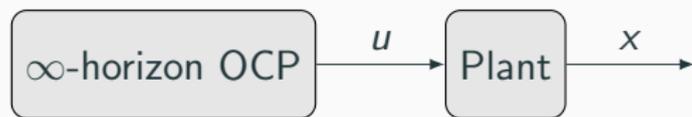
📄 Kalaria et al., 2025

📄 Le Cleac'h et al., 2022

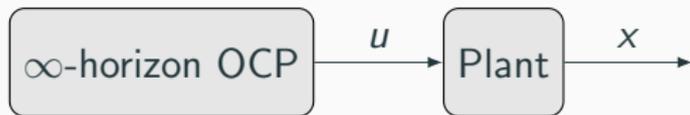📄 Fieni et al., 2025

📄 Baghbadorani et al., 2025

MPC in a nutshell

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- Constrained, non-LQ case intractable
- Open-loop operation

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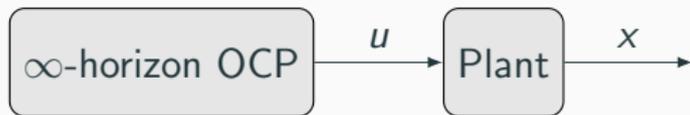
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MPC: Approximate via feedback
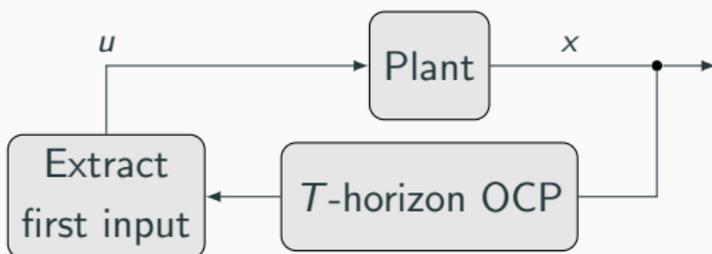
Game-theoretic MPC in a nutshell
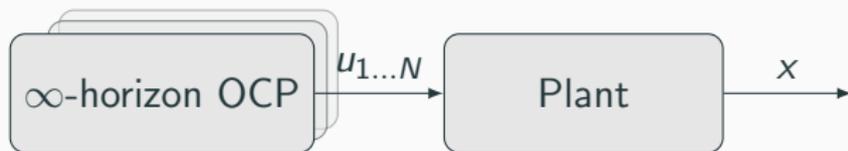
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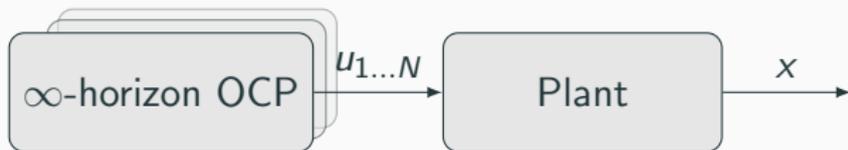
Game MPC
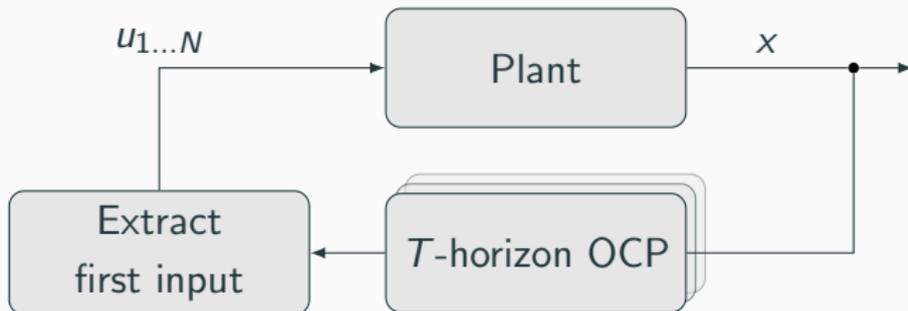
- How to design the finite-horizon OCPs to match the ∞-horizon performance?
- How can we compute the solution in real-time?

Infinite-horizon optimality

🎮 **Optimal control**	🎮
$x^{t+1} = Ax^t + Bu^t$ $J(u) = \sum\limits_{t=0}^{\infty} xQx + uRu$	

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Optimal control:	
$u^* \in \arg\min\limits_{u} J(u)$	

$\boxed{\infty\text{-horizon OCP}} \longrightarrow \boxed{\text{Plant}} \longrightarrow$

🎮 **Optimal control**	🎮 **Dynamic game**
$x^{t+1} = Ax^t + Bu^t$	$x^{t+1} = Ax^t + \sum\limits_i B_i u_i^t$
	For each agent i:
$J(u) = \sum\limits_{t=0}^{\infty} xQx + uRu$	$J_i(u_i, u_{-i}) = \sum\limits_{t=0}^{\infty} x^t Q_i x^t + u_i^t R_i u_i^t$
Optimal control:	
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∞-horizon OCP → Plant →

🎮 **Optimal control**	🎮 **Dynamic game**
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Optimal control:	open-loop Nash eq:
$u^* \in \arg\min_u J(u)$	$u_i^* \in \arg\min_{u_i} J_i(u_i, u_{-i}^*)$, for all i

Alg. Riccati equation

$$\begin{cases} P = Q + A^\top P(A + BK) \\ K = -R^{-1}B^\top P(A + BK) \end{cases}$$

$u^* = Kx$

Achieved objective:

$J^* = xPx$

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Coupled AREs

for each agent i:

$$\begin{cases} X_i = Q_i + A^\top X_i(A + \sum B_i K_i) \\ K_i = -R_i^{-1}B_i^\top X_i(A + \sum B_i K_i) \end{cases}$$

$u_i^* = K_i x_i$

📄 Monti et al., 2024

🎮 **Alg. Riccati equation**	🎮 **Coupled AREs**
$$\begin{cases} P = Q + A^\top P(A + BK) \\ K = -R^{-1}B^\top P(A + BK) \end{cases}$$ $u^* = Kx$	for each agent i: $$\begin{cases} X_i = Q_i + A^\top X_i(A + \sum B_i K_i) \\ K_i = -R_i^{-1}B_i^\top X_i(A + \sum B_i K_i) \end{cases}$$ $u_i^* = K_i x_i$
	📄 Monti et al., 2024
Achieved objective: $J^* = xPx$	**Lemma:**
	can construct P_i from X_i, K_i: $J_i^* = \begin{bmatrix} x \\ x \end{bmatrix} P_i \begin{bmatrix} x \\ x \end{bmatrix}$
	📄 Benenati and Grammatico, 2025

Agent i's model of the problem at the ARE solution $(K_{1,\dots,N})$

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Opponent's belief of state evolution – cannot be influenced

Agent i's model of the problem at the ARE solution $(K_{1,...,N})$

Opponent's belief of state evolution – cannot be influenced

Actual state evolution – can be influenced
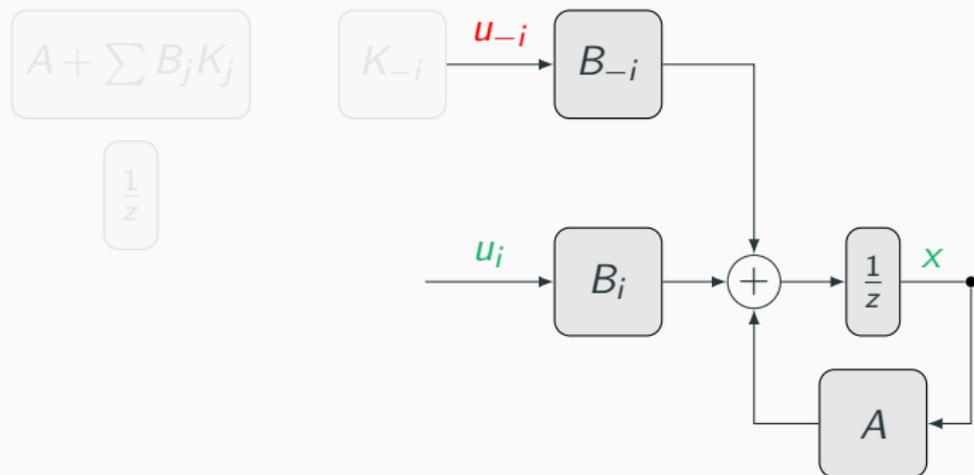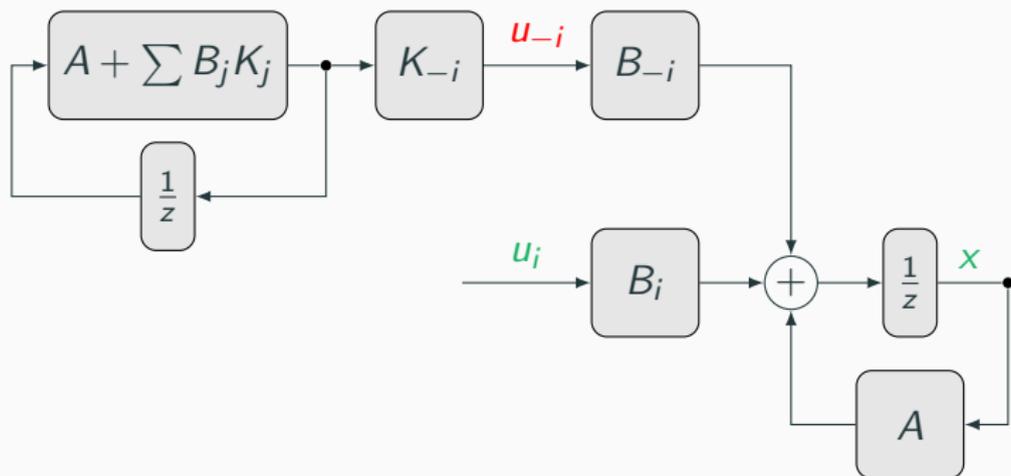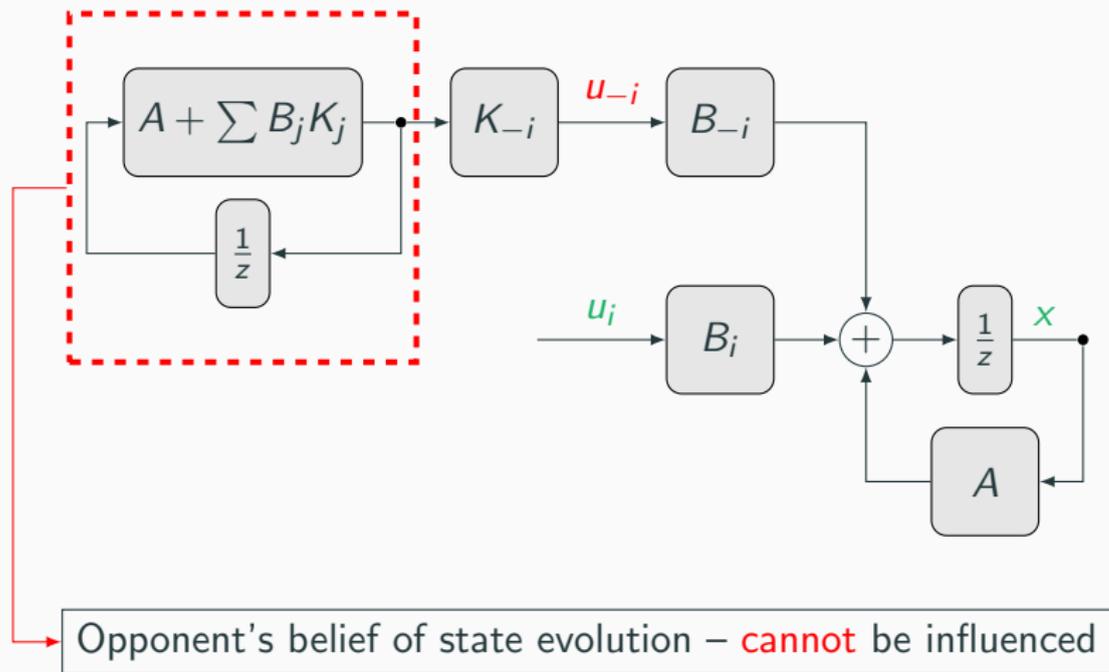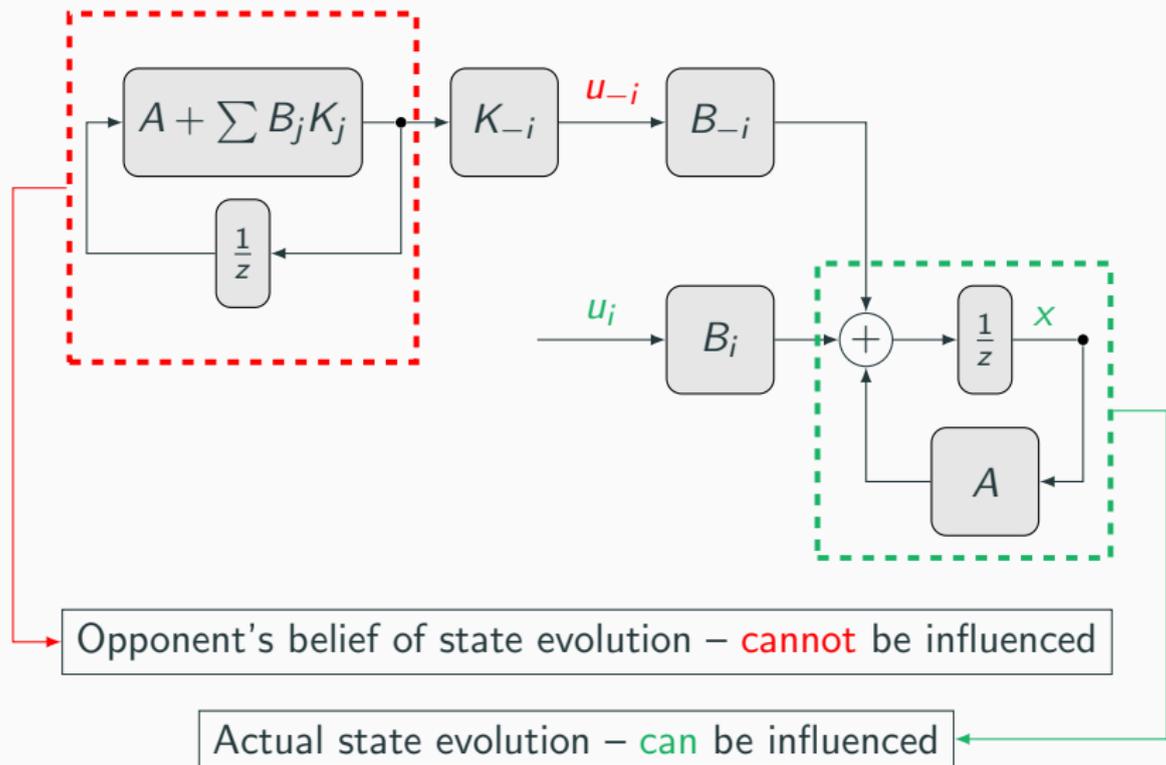
🎮 ∞-**horizon optimal MPC**	👥
$$u^* \in \arg\min_u \sum_{t=0}^{\infty} x^t Q x^t + u_i R u_i^t$$	

🎮 ∞-horizon optimal MPC	👥
$u^* \in \arg\min\limits_{u} x^N P x^N + \sum\limits_{t=0}^{N} x^t Q x^t + u_i R u_i^t$	

🎮 ∞-**horizon optimal MPC**	👥 ∞-**horizon Nash MPC**
$$u^* \in \arg\min_u x^N P x^N + \sum_{t=0}^{N} x^t Q x^t + u_i R u_i^t$$	
Theorem	
$\underbrace{u^{*1}, \dots, u^{*N}}_{\textit{horizon}}, \underbrace{Kx^N, Kx^{N+1}\dots}_{\textit{"terminal controller"}}$ is ∞-horizon optimal.	
📄 Bei Hu and Linnemann, 2002	

🎮 ∞-**horizon optimal MPC**	👥 ∞-**horizon Nash MPC**
$$u^* \in \arg \min_u x^N P x^N + \sum_{t=0}^{N} x^t Q x^t + u_i R u_i^t$$	Find u^* such that $$u_i^* \in \arg \min_{u_i} \sum_{t=0}^{\infty} x^t Q_i x^t + u^t R_i u^t$$ $$x^{t+1} = A x^t + B_i u_i^t + \sum_{j \neq i} B_j u_j^{t*}$$

Theorem

$$\underbrace{u^{*1}, \ldots, u^{*N}}_{\text{horizon}}, \underbrace{Kx^N, Kx^{N+1}\ldots}_{\text{``terminal controller''}}$$

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🎮 **∞-horizon optimal MPC**	👥 **∞-horizon Nash MPC**

Find u^* such that

$$u^* \in \arg\min_u x^N P x^N + \sum_{t=0}^{N} x^t Q x^t + u_i R u_i^t$$

$$u_i^* \in \arg\min_{u_i} \begin{bmatrix} x^N \\ \xi^N \end{bmatrix} P_i \begin{bmatrix} x^N \\ \xi^N \end{bmatrix} + \sum_{t=0}^{N} x^t Q_i x^t + u^t R_i u^t$$

$$x^{t+1} = A x^t + B_i u_i^t + \sum_{j \neq i} B_j u_j^{t*}$$

Theorem

$$\underbrace{u^{*1}, \ldots, u^{*N}}_{\text{horizon}}, \underbrace{K x^N, K x^{N+1} \ldots}_{\text{"terminal controller"}}$$

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🎮 ∞-horizon optimal MPC	🎮 ∞-horizon Nash MPC

∞-horizon optimal MPC

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∞-horizon Nash MPC

Find u^* such that

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$$x^{t+1} = A x^t + B_i u_i^t + \sum_{j \neq i} B_j u_j^{t*}$$

$$\xi^{t+1} = A \xi^t + B_i u_i^{t*} + \sum_{j \neq i} B_j u_j^{t*}$$

🎮 ∞-horizon optimal MPC	🎮 ∞-horizon Nash MPC
$u^* \in \arg\min_u x^N P x^N + \sum_{t=0}^{N} x^t Q x^t + u_i R u_i^t$	Find u^* such that $$u_i^* \in \arg\min_{u_i} \begin{bmatrix} x^N \\ \xi^N \end{bmatrix} P_i \begin{bmatrix} x^N \\ \xi^N \end{bmatrix} + \sum_{t=0}^{N} x^t Q_i x^t + u^t R_i u^t$$ $$x^{t+1} = A x^t + B_i u_i^t + \sum_{j \neq i} B_j u_j^{t*}$$ $$\xi^{t+1} = A \xi^t + B_i u_i^{t*} + \sum_{j \neq i} B_j u_j^{t*}$$
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📄 Bei Hu and Linnemann, 2002	📄 Benenati and Grammatico, 2025

LQ games as variational inequalities

🎮 MPC as QP	👥
Substitute out the state, derive: $\nabla_u J = Hu + Fx^0$	

🎮 MPC as QP	👥 Game MPC as VI
Substitute out the state, derive: $\nabla_u J = Hu + Fx^0$	
Nec. and suff. for optimality: **KKT of a QP**	
$0 = Hu + Fx^0 + C^\top \lambda$ $0 \leq \lambda \perp -Cu - Dx^0 + c \geq 0$	

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Substitute out the state, derive: $$\nabla_u J = Hu + Fx^0$$	$$\begin{bmatrix} \nabla_{u_1} J_1 \\ \vdots \\ \nabla_{u_N} J_N \end{bmatrix} = H \begin{bmatrix} u_1 \\ \vdots \\ u_N \end{bmatrix} + Fx^0$$
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🎮 **MPC as QP**	👥 **Game MPC as VI**
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Nec. and suff. for optimality: **KKT of a QP**	Sufficient for Nash: **KKT of a VI**
$0 = Hu + Fx^0 + C^\top \lambda$ $0 \leq \lambda \perp -Cu - Dx^0 + c \geq 0$	$0 = Hu + Fx^0 + C^\top \lambda$ $0 \leq \lambda \perp -Cu - Dx^0 + c \geq 0$ H not symmetric!

How to solve a VI

Iterative solvers

Operator splitting   📄 Mignoni et al., 2025   **moAVIso**

Newton step + line-search   📄 Zhu and Borrelli, 2024

Douglas-Rachford   📄 Baghbadorani et al., 2025

Interior point method   📄 Liu and Liao-McPherson, 2025

How to solve a VI

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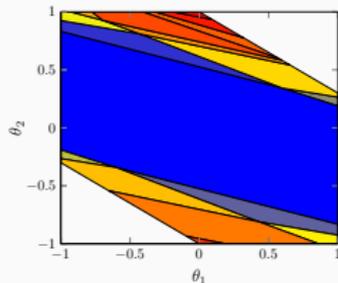
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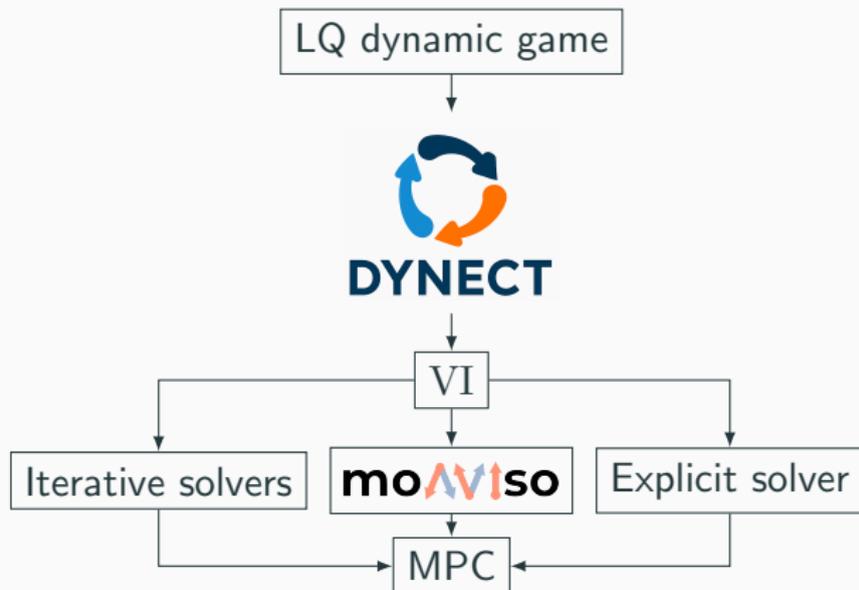
Interior point method   📄 Liu and Liao-McPherson, 2025

Explicit solution mapping

- Solution is piecewise affine in x^0

- One region per combination of constraints
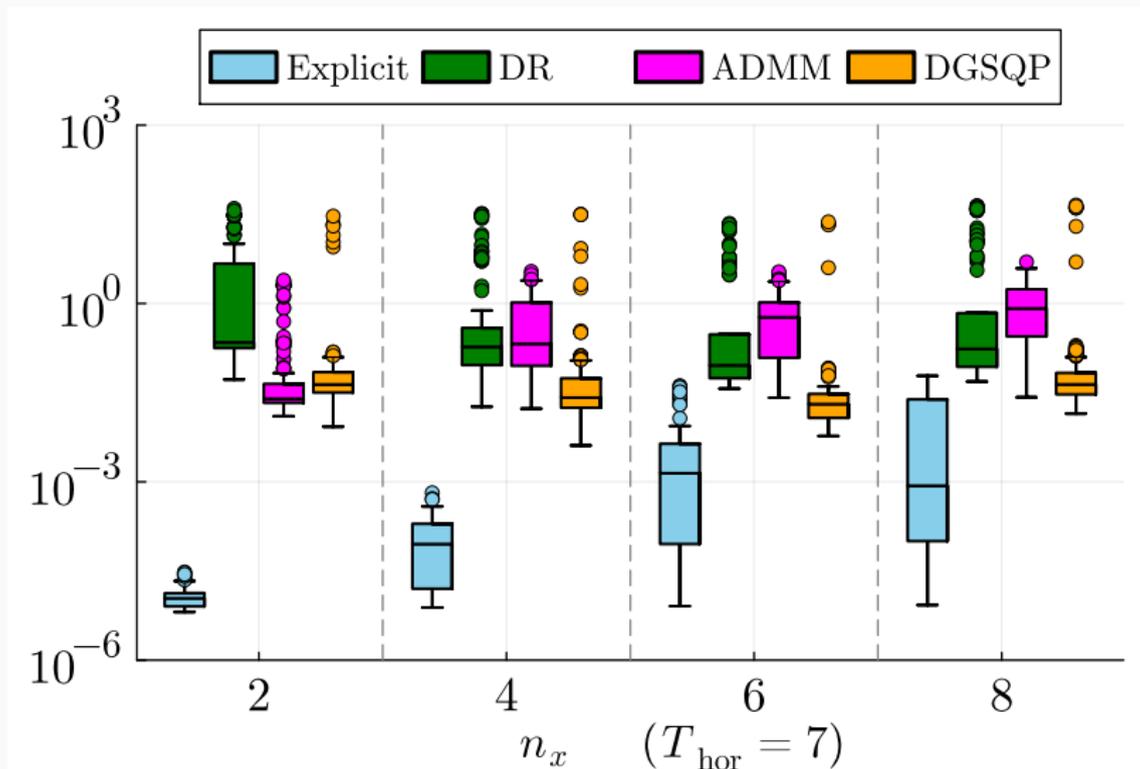
- IDEA: Compute offline

- New!   📄 arXiv:2512.07749
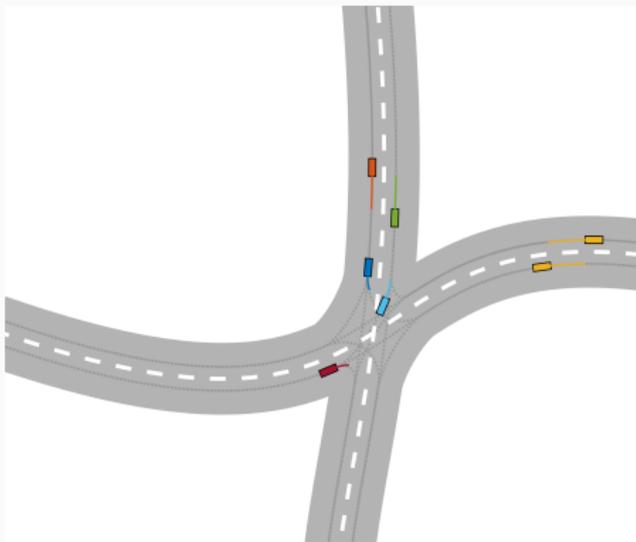
The Dynamic Nash Equilibrium Control Toolbox

Online solution time

Random LQ games, N agents $= 6$, $n_{\mathrm{u}} = 4$, Horizon $= 7$

Conclusion

Take-home messages

- Infinite-horizon constrained LQ games can be solved as a finite-horizon VI for some initial conditions
- Asymptotic stability of game MPC
- The solution is PWA in the initial condition
- Explicit solution computable offline
- Real-time computation is a reality

Thank you!

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📄 TAC 2026 (accepted)

📄 arXiv 2025

G. Belgioioso

S. Grammatico

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